# Lecture 8 - Sep 30

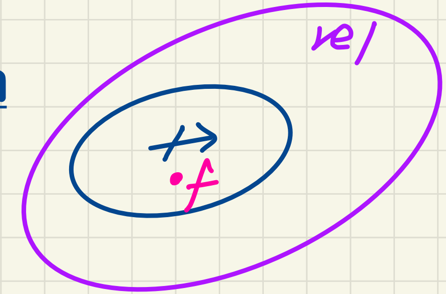
## Math Review

***Rel Image vs. Func Application***
***Modelling: Rel vs. Partial vs. Total Func***
***Injection, Surjection, Bijection***

# Announcements/Reminders

- Today's class: notes template posted
- Event-B Summary Document
- Priorities:
  - Lab1 → Review
  - Lab2 → Review
- Change of ProgTest venue – WSC106/108
- Released:
  - ProgTest guide
  - 2 Practice Tests and solutions
  - Lab1, Lab2 solutions

# Relational Image vs. Functional Application




**A function is a relation.**

$$\underbrace{f}\in \underbrace{\{1, 2, 3\}}_{S} \nrightarrow \underbrace{\{a, b\}}_{T}$$

IsFunction(f) ∧ dom(f) ⊆ {1, 2, 3}

dom(f) ⊂ {1, 2, 3} → at least one value from S that does not have the corresponding range value.

dom(f) = {1, 2, 3}

$$f = \{(3, a), (1, b)\}$$

relational range

## Exercises:

treating f as a relation

$f[\{3\}] = \{a\}$

$f[\{1\}] = \{b\}$

$f[\{2\}] = \emptyset$ (2 ∈ S, 2 ∉ dom(f))

**functional application**

$f(3) = a$

$f(1) = b$

$f(2) = \bot$ (undefined)

# In Rodin

① $f : \mathbb{Z} \leftrightarrow \mathbb{Z}$

$f[\{c\}]$ — always well-defined

$c \in \mathbb{Z}$

② $g : \mathbb{Z} \rightarrowtail\!\!\!\!+ \mathbb{Z}$

$g(c)$ — not always well-defined

↳ one PO generated: $c \in dom(g)$.

$c \in \mathbb{Z}$

## Cardinality of relational image

$r \in S \longleftrightarrow T$

$r$ is also a function

$\Rightarrow |r[\{s\}]| \leq 1$

$s \in S$

$= 1 \rightsquigarrow s \in dom(r)$

$= 0 \rightsquigarrow s \notin dom(r)$.

# Modelling Decision: Relations vs. Functions

An organization has a system for keeping **track** of its employees as to where they are on the premises (e.g., ``Zone A, Floor 23''). To achieve this, each employee is issued with an active badge which, when scanned, synchronizes their current positions to a central database.

Assume the following two sets:

- *Employee* denotes the **set** of all employees working for the organization.
- *Location* denotes the **set** of all valid locations in the organization.

Is where_is ∈ Employee <-> Location appropriate?

No. $e1 \mapsto l_1 \in where\_is \land e1 \mapsto l_2 \in where\_is$

Is where_is ∈ Employee → Location appropriate?

↳ No ∵ some employees may not be in the company.

Is where_is ∈ Employee ⇸ Location appropriate?

YES.

# Functions

| | (dom) injective | (ran) surjective | (dom, ran) bijective |
|---|---|---|---|
| partial $\nrightarrow$ | partial injection $\rightarrowtail\!\!\!\!\!+$ | partial surjection $\nrightarrow\!\!\!\!\!\twoheadrightarrow$ | n.a. |
| total $\rightarrow$ | $\rightarrowtail$ | $\twoheadrightarrow$ | $\rightarrowtail\!\!\!\!\!\twoheadrightarrow$ |

** Contrapositive: $s1 \neq s2 \Rightarrow \neg((s1,t) \in f \wedge (s2,t) \in f)$

# Injective Functions

func. property

isInjective(f)

$\forall s, t1, t2 \bullet (s \in S \wedge t1 \in T \wedge t2 \in T) \Rightarrow ((s,t1) \in f \wedge (s,t2) \in f \Rightarrow t1 = t2)$

$\forall s_1, s_2, t \bullet (s_1 \in S \wedge s_2 \in S \wedge t \in T) \Rightarrow ((s_1, t) \in f \wedge (s_2, t) \in f \Rightarrow s_1 = s_2)$

inj. property

same range value

cannot have two distinct dom. values mapping to the same range value.

If $f$ is a ***partial*** ***injection***, we write: $f \in S \nrightarrowtail T$

- e.g., $\{\ \varnothing, \{(1, \mathbf{a})\}, \{(2, \mathbf{a}), (3, \mathbf{b})\}\ \} \subseteq \{1, 2, 3\} \nrightarrowtail \{a, b\}$
- e.g., $\{(1, \mathbf{b}), (2, a), (3, \mathbf{b})\} \notin \{1, 2, 3\} \nrightarrowtail \{a, b\}$
- e.g., $\{(1, \mathbf{b}), (3, \mathbf{b})\} \notin \{1, 2, 3\} \nrightarrowtail \{a, b\}$

If $f$ is a ***total*** ***injection***, we write: $f \in S \rightarrowtail T$

- e.g., $\{1, 2, 3\} \rightarrowtail \{a, b\} = \varnothing$
- e.g., $\{(2, d), (1, a), (3, c)\} \in \{1, 2, 3\} \rightarrowtail \{a, b, c, d\}$
- e.g., $\{(\mathbf{2}, d), (\mathbf{1}, c)\} \notin \{1, 2, 3\} \rightarrowtail \{a, b, c, d\}$
- e.g., $\{(2, \mathbf{d}), (1, c), (3, \mathbf{d})\} \notin \{1, 2, 3\} \rightarrowtail \{a, b, c, d\}$

e.g. $f = \{(1,b), (2,b)\}$ ($s_1$, $t$, $s_2$, $t$)

$((1,b) \in f \wedge (2,b) \in f) \Rightarrow 1 = 2$ false.

func(∀) prop. ∧ injective(∀) prop.

If $f$ is a ***partial injection***, we write: $f \in S \mathrel{+\!\!\!\!\rightarrowtail} T$

- e.g., $\{\varnothing, \{(1,\mathbf{a})\}, \{(2,\mathbf{a}),(3,\mathbf{b})\}\} \subseteq \{1,2,3\} \mathrel{+\!\!\!\!\rightarrowtail} \{a,b\}$
- e.g., * $\{(1,\mathbf{b}),(2,a),(3,\mathbf{b})\} \notin \{1,2,3\} \mathrel{+\!\!\!\!\rightarrowtail} \{a,b\}$ * func ✓ inj X ∵ distinct dom values 1 and 2 both map to b.
- e.g., $\{(1,\mathbf{b}),(3,\mathbf{b})\} \notin \{1,2,3\} \mathrel{+\!\!\!\!\rightarrowtail} \{a,b\}$ X

If $f$ is a ***total injection***, we write: $f \in S \rightarrowtail T$

- e.g., $\{1,2,3\} \rightarrowtail \{a,b\} = \varnothing$
- e.g., $\{(2,d),(1,a),(3,c)\} \in \{1,2,3\} \rightarrowtail \{a,b,c,d\}$
- e.g., $\{(\mathbf{2},d),(\mathbf{1},c)\} \notin \{1,2,3\} \rightarrowtail \{a,b,c,d\}$
- e.g., $\{(2,\mathbf{d}),(1,c),(3,\mathbf{d})\} \notin \{1,2,3\} \rightarrowtail \{a,b,c,d\}$

sat. both func. & inj. properties (∀) trivially

distinct dom values map to distinct ran values ⇒ injective.

$\{(\underline{1}, \underline{a}), (\underline{2}, \underline{a})\}$

↳ function

not injective

$\{(1, a), (1, b)\}$

↳ **not** a function!

$S \leftrightarrow T$ : set of all relations

→ set of all possible total injections.

If $f$ is a ***total injection***, we write: $f \in S \rightarrowtail T$

- e.g., $\{1, 2, 3\} \rightarrowtail \{a, b\} = \emptyset$
- e.g., ① $\{(2, d), (1, a), (3, c)\} \in \{1, 2, 3\} \rightarrowtail \{a, b, c, d\}$
- e.g., ② $\{(2, d), (1, c)\} \notin \{1, 2, 3\} \rightarrowtail \{a, b, c, d\}$
- e.g., ③ $\{(2, \mathbf{d}), (1, c), (3, \mathbf{d})\} \notin \{1, 2, 3\} \rightarrowtail \{a, b, c, d\}$

witness

$\{(1, a), (2, b), (3, a)\}$

↓ violates inj. prop

| | func. prop | total | inj. prop |
|---|---|---|---|
| ① | ✓ | ✓ | ✓ |
| ② | ✓ | ✗ | ✓ |
| ③ | ✓ | ✓ | ✗ |

# Surjective Functions

$f \in S \leftrightarrow T$

$total(f) \iff dom(f) = S$

$$isSurjective(f) \iff ran(f) = T$$

① funct prop.
② surj prop.

If $f$ is a ***partial surjection***, we write: $f \in S \twoheadrightarrow\!\!\!\!\!+ \; T$

- e.g., $\{ \{(1,\mathbf{b}),(2,\mathbf{a})\}, \{(1,\mathbf{b}),(2,\mathbf{a}),(3,\mathbf{b})\} \} \subseteq \{1,2,3\} \twoheadrightarrow\!\!\!\!\!+ \; \{a,b\}$
- e.g., $\{(2,\mathbf{a}),(1,\mathbf{a}),(3,\mathbf{a})\} \notin \{1,2,3\} \twoheadrightarrow\!\!\!\!\!+ \; \{a,b\}$
- e.g., $\{(2,\mathbf{b}),(1,\mathbf{b})\} \notin \{1,2,3\} \twoheadrightarrow\!\!\!\!\!+ \; \{a,b\}$

→ func not surj.

If $f$ is a ***total surjection***, we write: $f \in S \twoheadrightarrow T$

- e.g., ① $\{ \{(2,a),(1,b),(3,a)\}$, ② $\{(2,b),(1,a),(3,b)\} \} \subseteq \{1,2,3\} \twoheadrightarrow \{a,b\}$
- e.g., ③ $\{(2,a),(3,b)\} \notin \{1,2,3\} \twoheadrightarrow \{a,b\}$
- e.g., ④ $\{(2,\mathbf{a}),(3,\mathbf{a}),(1,\mathbf{a})\} \notin \{1,2,3\} \twoheadrightarrow \{a,b\}$

| | total | func | surj. | |
|---|---|---|---|---|
| ① | ✓ | ✓ | ✓ | ✓ |
| ② | ✓ | ✓ | ✓ | ✓ |
| ③ | ✗ | ✓ | ✓ | ✗ |
| ④ | ✓ | ✓ | ✗ | ✗ |

# Bijective Functions

$f$ is ***bijective***/***a bijection***/*one-to-one correspondence* if $f$ is ***total***, ***injective***, and ***surjective***.

all possible bijections

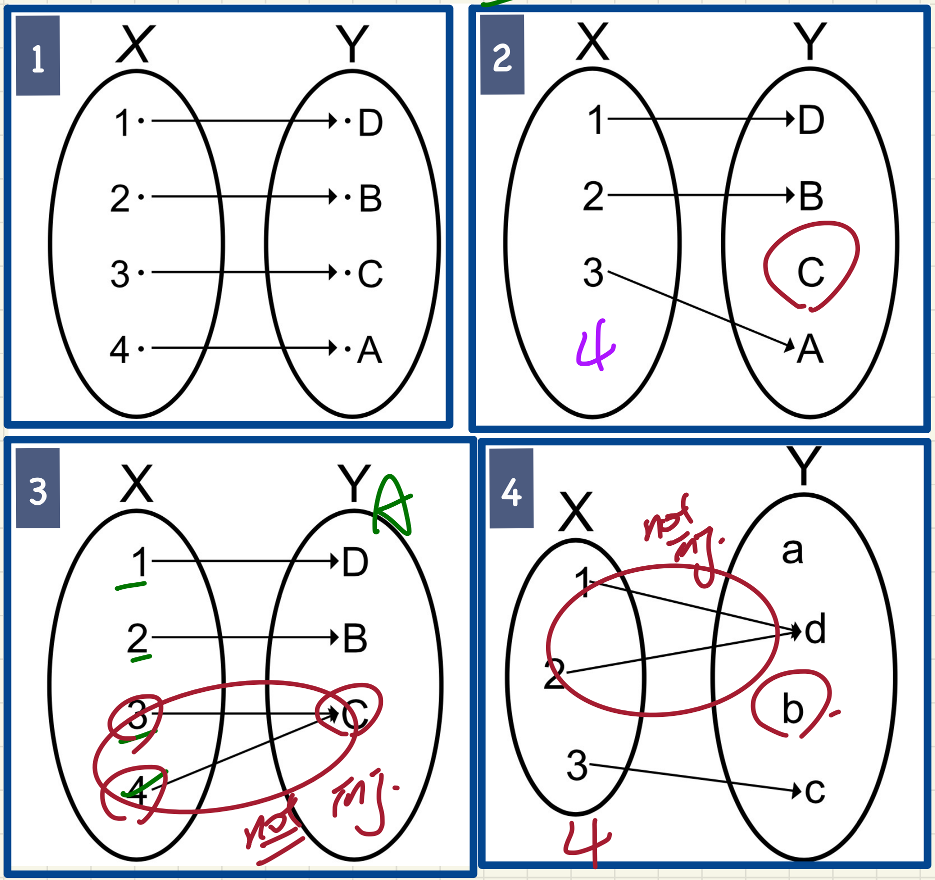
- e.g., $\{1,2,3\} \rightarrowtail\!\!\!\!\twoheadrightarrow \{a,b\} = \varnothing$ → ∵ no injective function can be made
- e.g., $\{\{(1,a),(2,b),(3,c)\},\{(2,a),(3,b),(1,c)\}\} \subseteq \{1,2,3\} \rightarrowtail\!\!\!\!\twoheadrightarrow \{a,b,c\}$
- e.g., ① $\{(\mathbf{2},b),(\mathbf{3},c),(\mathbf{4},a)\} \notin \{1,2,3,4\} \rightarrowtail\!\!\!\!\twoheadrightarrow \{a,b,c\}$
- e.g., ② $\{(1,\mathbf{a}),(2,b),(3,c),(4,\mathbf{a})\} \notin \{1,2,3,4\} \rightarrowtail\!\!\!\!\twoheadrightarrow \{a,b,c\}$
- e.g., ③ $\{(1,\mathbf{a}),(2,\mathbf{c})\} \notin \{1,2\} \rightarrowtail\!\!\!\!\twoheadrightarrow \{a,b,c\}$

| | total + func | inj | surj. | |
|---|---|---|---|---|
| ① | ✗ | ✓ | ✓ | ✗ |
| ② | ✓ | ✗ | ✓ | ✗ |
| ③ | ✓ | ✓ | ✗ | ✗ |

# Exercise

$X = \{1, 2, 3, 4\}$

$Y = \{A, B, C, D\}$




| | 1 | 2 | 3 | 4 |
| --- | --- | --- | --- | --- |
| partial | ✓ | ✓ | ✓ | ✓ |
| total | ✓ | ✗ | ✓ | ✗ |
| injection | ✓ | ✓ | ✗ | ✗ |
| surjection | ✓ | ✗ | ✗ | ✗ |
| bijection | ✓ | ✗ | ✗ | ✗ |